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LANGUAGE OF LETTERS IN ELEMENTARY MATHEMATICS AS TOOL FOR DEVELOPMENT OF THINKING

Dagmar Málková and Darina Jirotková ☜

Abstract

The paper describes the first results of a research study focusing on the process of generalization in primary school pupils. The research was conducted in two stages – individual solving of problems by pupils, analysis of pupils' solutions and selection of pupils for the second stage of the research which had the form an individual task-based interviews. The research tool was a set of different kinds of tasks of graded difficulty that allowed easy modification for individual pupils. By analysing the solution of two series of tasks by 3rd graders of primary school we can state that with appropriate formulation of tasks some pupils are able to formulate ideas at a general level and use the language of letters.

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Keywords: pre-algebra, theory of generic model, task-based interview, task design, language of letters

Introduction

Recently, a clearly formulated concept of mathematical literacy (ML) has been adopted by decision-makers in education in the Czech Republic such as the Ministry of Education (MEYS) and the Czech School Inspectorate (CSI). In consequence, this new definition asks for revisions of curricular documents. The core and binding document for all schools issued by the Ministry of Education is the Framework Educational Programme (FEP). It is the starting point for each school to create its own school educational programme, which adapts the FEP to the conditions and possibilities of the school.

Out of the seven items of ML we select and present here the three which are directly connected to the topic of this paper. They are:

1. The ability to understand different types of mathematical text (symbolic, verbal, pictorial, graphs, tables, letters) and to use actively or even form different mathematical languages.
2. The ability to gain and classify experience through one's own manipulative, speculative and inquiry-based activities, most often by the trial and error method.
3. The ability to generalize the gained experience, discover patterns and regularities, formulate hypotheses.

Obviously, language plays an important role in every area of our lives. It is a tool both for communication and the development of thinking. More accurate thinking enforces better communication. It is language that allows us to grasp and structure concepts and relationships, allows us to formulate problems and strategies for their solution, i.e. it enables the development of thinking in general and in mathematics in particular. One of the frequently used and very important languages in mathematics is the language of letters. It has several functions in primary school mathematics: coding function (solving problems using letters), transformational function (modifying relations in such a way that we find the unknown in an equation), expressive function (describing the unknowns and the relationships between them) and grasping function (allowing us to describe, grasp a certain set of numbers, even an infinitive set). (Hejný, 2014)

If pupils are to use the language of letters with understanding, it is necessary they meet it through situations that create the need of its introduction. One of the ways is through tasks where the pupil first gets a set of partial results whose appropriate organization (table, arrangement, graph) make it possible to derive some regularity or common characteristics which is then generalized, that is described in a way that is no longer bound to the specific subjects, e.g. using the language of symbols or letters.

Our experience shows it is important to solve tasks aiming at gradual generalization frequently with pupils. This helps them build metacognitive strategies for grasping these tasks. When solving these tasks, pupils develop, among other things, the ability to reflect on the acquired experience, compare, classify and look for common and different phenomena and also to formulate common characteristics of a group of phenomena. In other words, pupils look for a suitable representation of the phenomenon, which may have only secondary role in other tasks (Pytlak, 2011). This generalized knowledge gradually develops to the level of abstract knowledge (Hejný, 2012) through the process of desemantization (Hejný, Jirotková & Slezáková, 2015) and abstraction.

Current state

In FEP for Elementary Education (MŠMT, 2017), algebra is included in the thematic area Number and Variable for lower secondary level. Neither propaedeutic to algebra, nor the requirement to develop the ability to work with letters are included in any of the thematic areas for primary level. This means it is up to the authors of textbooks whether and how they include this topic in their textbooks.

In 2000, NCTM published Standards that present a reform conception of teaching mathematics in the USA and Canada and have had a major impact on development of didactics of mathematics all around the world (Hendl, 2002). A significant idea presented in the Standards is that algebraic thinking can be developed in mathematics lessons already at primary school level. Also research shows that this is crucially important (e.g. Kieran, 1992; Linchevski, 1995). Research confirms that if we start developing algebraic thinking on lower secondary school level, it is too late. An important role in it is played by connection of algebra and geometry as a tool for visualization of arithmetic phenomena. (NCTM, 2020)

Problems for lower secondary school pupils in the area bordering on algebra are documented for example in the results of the international survey TIMSS. In 2007, the problem with square windows was used for the 8th graders (M29 (M05-03)) (Tomášek et al., 2009). This problem was included in the topic Algebra – Lines and Sequences. The goal of the problem was to find the number of windows from 73 matches, i.e. to discover the relation between the number of windows and the number of matches and to express it in such a way that it would allow to determine the number of windows from a given number of matches.

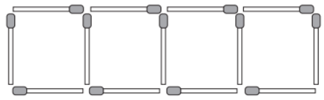

Figure 1: Square windows

The result was alarming. Only 8.8% of Czech pupils participating in the TIMSS survey were able to solve this problem correctly. The international average of correct answers was 8.7%. Obviously, not even 8th graders were able to handle this task. More than a fifth of the pupils did not even try to solve the problem. Clearly, the pupils did not see drawing a picture of windows from 73 matches as a possible strategy.

Theoretical framework

The area of algebra is difficult to understand for pupils due to its high abstractness (Prendergast & Treacy, 2018). Therefore, many mathematics educators look for a way to start developing algebraic thinking already on primary school level. Blanton and Kaput (2004), Carraher et al. (2006), Kieran et al. (2016) confirm that inclusion of algebra in lessons from an early age is beneficial for pupils. In her research, Pytlak (2011) states that in order to understand and use algebra correctly, it is necessary to master the basics of arithmetic. Primary school teachers in research of Rendl, Vondrová et al. (2013) state that the cause of pupils' failure to work with an unknown are that they are ontogenetically unprepared.

If we want to prepare a problem for pupils that will lead to the need to generalize and use symbols or letters, it is essential to understand cognitive processes in mathematics. The theory of the generic model (TGM), which describes the process of gaining knowledge, seems to be a suitable theoretical basis for this. "It starts with motivation and has as its core two mental shifts: the first leads from concrete knowledge (isolated models) to generalized knowledge (generic knowledge) and the second from generic to abstract knowledge." (Hejný, 2012, p. 44). TGM is a suitable tool also for such analysis of pupils' solutions whose goal is to uncover the pupils' thinking processes, to describe the identified error and its cause and also to set up further re-educational and educational processes in a way that the pupils' thinking is in the zone of approximate development (Brown & Ferrara, 1985).

Research questions

The goal of our research was to find suitable problems that would support the development of algebraic thinking of primary school pupils. Obviously, the problems had to be adapted in a way to be appropriate for this age category, i.e. pupils would be able to solve them with less effort. In other words, the problem would be in the zone of the pupil's approximate development. The problem was then set to pupils of one class and pupils' solutions were analysed. We tried to describe the level of each pupil on their way to abstraction, their ability to generalize, or use symbols or letters as a variable. Our research questions were:

- How did the 3rd graders of primary school solve the problem?
- Can we find an attempt at a generalized notation in the pupil's solution?

Methodology

The research was conducted in two stages. The first stage was conducted in presence form with 19 pupils of one class in September and October 2020. The tool of the experiment was a problem from wooden stick geometry.

The research database from this stage consists of 19 written pupils' solutions. 6 respondents for the second stage were selected based on our analysis of these solutions. The second stage of the research was conducted during distance education. The research tool was a problem following from the first problem oriented at generalization and individual interview of the experimenter with the pupil about the solution. The research database from the second stage consists of 6 video recordings that were transcribed and supplemented by the pupils' written solutions.

When selecting the problems as our research tool, we chose the following task design. The task difficulty was graded so that each pupil could solve the first of a series of tasks quite easily but would have to exert more effort over the last one – either mentally or to work harder. The greatness of the number was selected to be the gradation parameter. In the first part, it was the number of the wooden sticks, in the second, the number of windows that was given.

The second stage of the research used the method of a task-based interview with 6 selected pupils. The criterion for the selection of these pupils was that their solution of the first problem was somehow interesting and bore the potential of generalization under careful guidance of the experimenter.

Didactical analysis of the problems

Problem 1 for the first part of the experiment:

We can make 3 square windows from ten wooden sticks. Find how many windows we can create from: a) 19, b) 31, c) 100 wooden sticks?

The problem comes from the area of wooden stick geometry and connects arithmetic and geometry. The goal is to discover the relationship between the number of square windows and the number of wooden sticks needed for their construction ($(s\text{-}1):3 = w$, where w is the number of windows and s the number of wooden sticks). The problem assumes that the pupil has a good idea of the verbally described situation. Given that the pupils-respondents had already had some experience with wooden stick tasks at the time of the research, we could expect there would be no problem with the initial ideas. The first task can be solved manipulatively, but also easily only mentally. The second task can be easily solved graphically. The graphical solution of the third task is demanding, which creates pressure on choosing a more effective strategy, such as use of arithmetic. One of the possibilities is to record the process of creating a series of windows in a table. This gives two sequences – the number of bars and the number of windows. After detecting the regularity, it is possible to generalize the relationship for any number of wooden sticks.

The number of windows	1	2	3	4	5	6	
The number of wooden sticks	4	7	10	13	16	19	22 .....

Problem 2 for the second part of the experiment:

We know from problems we have been solving that we can make 10 windows out of 31 wooden sticks. How many sticks do we need to construct: a) 20 windows; b) 35 windows?

Tell me any number of windows and I will tell you how many sticks will be needed for their construction. Can you uncover the magic spell? Can you explain it?

This problem changes the direction of inquiry – we determine the number of wooden sticks from the number of windows (this relationship can be formulated e.g. $s = w \cdot 3+1$). The offer of "magic spell", which is based on the given relationship, should have a motivating effect as pupils try to formulate the relationship somehow and also collect more specific cases, or in other words isolated models of future knowledge (Hejný, 2012).

19 third graders of a Prague primary school were involved in the first part of the experiment. The pupils solved the problem independently in the environment of their classroom. They had 30 minutes to solve it. After a whole class discussion on the used solving strategies and results, the written solutions were collected from the pupils. These made the first part of the research database.

All solutions were divided into three groups – the first group were solution with only arithmetic solution without and drawing. This group consisted of 6 solutions. The second group included purely graphical solutions. There were 11 of them. The remaining two solutions were a combination of graphical solution and calculation. The solutions in each of these groups were further divided according to additional criteria such as correctness, completeness, presence of the idea of generalization.

Six pupils were selected for the second stage of the research from each of the three groups. Due to the scope of the paper, we focus here only on two solutions. It is the solution of the pupil Adam that represents the subgroup Correct arithmetic solutions, complete, with a trace of the thought of generalization. The solution of the pupil Bára represents the subgroup Graphical solutions, incomplete, with a trace of the thought of generalization.

The pupils solved the second problem out loud if possible and the experimenter was allowed to ask questions. The problem was formulated in such a way which would open up the possibility of generalizing and also would explore the pupils' ability of prealgebraic thinking. In the section, we will present interviews with only 2 pupils due to the limited scope of the article.

Pupil Adam – in his solutions there are hints of generalization. He solved the task arithmetically, he does not feel the need to visualize the situation, he connects adding a square to adding number 3. We were interested if the pupil used the result from task a) in the solution of task b). His record shows this as probable.

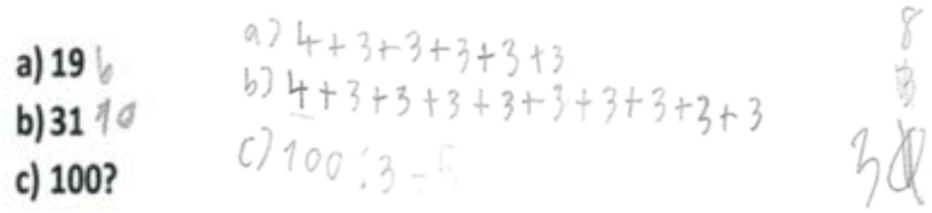

Figure 2: Adam’s solution

Pupil Bára – starts solving graphically but then feels the need to describe the strategy – using multiplication for 3. She makes a grammar mistake (po mocí should be one word, it is not a preposition and a noun). She mentions multiplication by 3 twice. This could suggest she is aware of some relationships between the number of wooden sticks and windows.

Figure 3: Bára’s solution

The second stage of the experiment was conducted online due to distance education in the time of COVID 19 pandemic. The interviews with the pupils were recorded and then transcribed and analysed. We also collected the pupils’ written solutions.

The pupils were contacted individually. At first, the pupils solved the problem independently for a while so that no one intervene with their thinking. Having solved the problem, interviews were conducted about the solving method. At the same time we tried to ask questions that would guide pupils to some general notation or some (pre)algebraic notation.

Let us now present the relevant statements from interviews with pupil Adam and pupil Bára.

Pupil Adam: Adam is very good at numeration. He likes solving logical problems. He was excited about taking part in the research. He solved the task within 3 minutes. He calculated and drew nothing. He explained:

> A01: So in the first one, there are twenty windows, and I took twenty times three. Because for one window we need those three and at the beginning there is that one, so that it makes the three. But then I did twenty times three is sixty plus one, that one at the very beginning, so that is sixty one. Exactly the same.

TA01: So again twenty times three plus one?

A02: No. Thirty five times three is one hundred and fifteen plus one is one hundred and sixteen.

Having been drawn attention to it, Adam immediately reacted to his mistake and realized on his own where the mistake had been made:

A03: But three times thirty is ninety and then three times five is fifteen. Aha! In the beginning I wrote one hundred and five but then one hundred and fifteen.

The pupil needs no scaffolding in visualization. He transfers the geometrical situation into numbers and thinks in the area of arithmetic.

TA03: I'd like to know if you could substitute the numbers twenty, thirty five, one hundred and five, three by something else so that I could then substitute any number that I would multiply by three, add one and got the result – the number of wooden sticks.

A04: Simple a different task only the initial number will be different.

...

TA08: So if I told anyone what you've just come up with, they would be able to calculate the number of wooden sticks?

A09: Yes. Unless they have a different shape. Then the three must be replaced by a different number.

...

A11: Windows times two plus one is something.

TA11: Great. And if we wanted to make it shorter, can you think for some abbreviation for windows?

A12: W.

In this situation, Adam's mathematical knowledge at the stage of a generic model – he was able to generalize several specific cases (isolated models) and formulate the relationship in words (A11). He could take the step towards abstract knowledge and formulation of the sought relationship if encouraged: "Write down what you have said". In our experience, pupils first write everything in words but in a short time they feel the need to shorten the words, even to one letter. Adam suggested that the word window could be abbreviated to W. This means Adam is a small step from entering algebra and using letters in formulating a given relationship, i.e. a small step from abstract knowledge. It is very interesting that Adam realizes that the relationship is valid only if the windows have the shape of a square (A09). He is also aware of the fact that the shape of the square is given by the presence of number three in the described relationship.

Pupil Bára: She solves the problem with the help of a drawing. She has low self-confidence – she is afraid to explain her solutions in order not to make a mistake. She was happy about her participation in the research, she likes in pair with a teacher. When drawing, she counted the sticks one by one and always made a dot. The interview with Bára was quite demanding. She could not describe her thinking processes. The experimenter asked Bára questions in such a way as to provoke the need to count the wooden sticks and guide her to an arithmetic understanding of the situation. In the end, Bara said that she would always solve this type of problem by drawing because it would be easiest for her, and that if she was to calculate it, it would be very difficult for her: “I don't know. Because it would be very difficult.”

Bára’s knowledge is at the first stage of the cognitive process – isolated models. She senses that it could be solved in other ways, but she feels no need for that: “I would solve it by drawing anyway, because this way is easiest for me.”

Conclusions

The article presents two analyses of pupils’ solutions and extracts from transcriptions of interviews with two pupils. These analyses allow us to conclude that on primary level, there are some 3rd graders who have the ability to make a generalized record using a letter. Adam is an example of this and thus confirms conclusions from other research (Carraher et al., 2006; Pytlak, 2011). Adam shows that with the right formulation of tasks he can progress in his thinking through generalization to abstraction where he formulates his ideas on a general level, and he could possibly record them in the language of letters. Motivation of the pupil to make this progress plays an important role in this process. The effort is then often crowned with the joyful cry “I got it!”, which testifies that the aha-effect has occurred. The pupil Bára did not have this motivation. She was not yet “mature” enough to be able to take the generalizing step. Bára belongs to the careful and average performing pupils in her class.

We have presented the type of problems that are suitable for initiation of the generalization process. An important parameter of these problems is that they are graded. Gradation needs to be done in such a way that the set of tasks starts with a task that will be appropriate for weaker pupils both with respect to its mathematical core and the possible solving strategies. This allows also the weaker pupils to fulfil the goal – they will solve the task. The last task in a graded series can then concern only a few individuals who, for example, with a little help from the teacher, can take a step towards using letters.

Inspired by the research of Slavíčková (2021), in which the author focuses on comparing pupils’ and pre-service teachers’ solutions, we decided to set a similar problem to pre-service primary school teachers at the Faculty of Education, Charles University. It was also graded, with the difference that in task c) letter *n* was used in the assignment. The students tried to use the letter from the very beginning. No one used the strategy of gradual generalization: to start from concrete experience and its possible enrichment through the process of generalization with the subsequent further cognitive lift – abstraction to express the desired relationship algebraically. Students have not yet encountered this method of solving algebraically formulated problems, and therefore were looking for solutions that they learned at upper secondary school level.

The presented research is qualitative, conducted with a small research sample. It does not allow us to make great conclusions but illustrates well that this type of problems and tasks supports significantly development of thinking.

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DISCOVERING MATHEMATICS IN THE MUSEUM OF AFRICAN ART

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Abstract

The newest reform of the educational system in Serbia brings forward the project approach on all levels of the educational system. In the preschool program, it is advocated as a dominant approach. This paper describes one curricular experiment – implemented at the University of Belgrade in Serbia that has enabled a group of prospective preschool teachers to implement the project approach. Our research project aimed to document the preservice preschool teachers' competencies to use the project approach. The experimental program explored The Museum of African Art as a context for discovering mathematics ideas which were later used in project activities with preschool children. Here we discuss findings based on a sample of 46 interviews conducted with prospective preschool teachers who were asked to reflect on their experience. The analysis shows that the Museum presented a valuable cultural resource for learning and student teachers developed partial sensitivity to opportunities for learning mathematics.

Keywords: integrative learning, project approach, preschool, teacher training

Introduction

The way we educate prospective preschool teachers in the Methodology course reflects the way we expect them to work on the development of mathematics concepts for children at preschool. At the beginning of our discussion, let us define major principles. We operate from a social constructivist position (von Glaserfeld, 1985). Development of mathematics thinking and building conceptual

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