DEVELOPING AN INTEGRATED FRAMEWORK FOR ANALYZING WAYS OF REASONING IN MATHEMATICS

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Abstract

Mathematics education literature involves studies that sought a way of investigating the mode of reasoning in mathematics textbooks because textbooks are the main resource for teachers in planning their mathematics lessons. In this vein, this study aimed to analyze the ways of reasoning in mathematics textbooks that are currently used in five countries: Slovakia, Czech Republic, Italy, Norway, and Turkiye, as a part of a Horizon 2020 Project. We initially started with a framework that aimed to examine the effect of teachers' participation in the lesson study on the improvement of students' mathematical reasoning (Project LESSAM). However, as the textbook analysis of different countries proceeded, we realized that this framework would not solely be sufficient to address the types of reasoning with alternative tasks presented in those textbooks. Thus, our framework development followed four main steps: (1) starting with a proposed framework that focused on different types of reasoning (2) identification of reasoning and proof tasks among worked examples, (3) categorization of ways of reasoning through the proposed framework, and (4) developing new categories based on a review of existing frameworks, to cover and differentiate all types of reasoning in the worked examples. Hence, in this study, we aim to present the integrated framework that we developed to analyze the ways of reasoning in worked examples (the problems with explained solutions) in the above-mentioned countries' textbooks. We also discuss future research agenda for analyzing mathematics textbooks with this integrated framework.

Keywords: Mathematics education, ways of reasoning, mathematics worked examples, mathematics textbook analysis.

1 INTRODUCTION

Doing mathematics is based on reasoning that may be enacted in different ways such as deductive and inductive reasoning [1] Therefore, mathematics education researchers put their attention to identifying different modes of reasoning that textbook problems conveyed (e.g., [1], [2], [3]). In those studies, the researchers not only examined the Reasoning and Proof (R&P) tasks in textbooks but also aimed to classify ways of argumentation and reasoning involved in the tasks. For instance, Stacey and Vincent [1] argued that the main purpose of explanations was rule derivation, rather than using explanations as thinking tools to develop justifications. In another study, Stylianides [4] focused on R&P tasks and developed a framework that could be used as an analysis tool in textbook analysis and as an instructional tool in teacher professional development sessions. In this framework, he particularly looked for the ways of making mathematical generalizations and providing support to mathematical claims. Utilizing this framework, Bieda et al. [2] analysed seven 5^{th}-grade mathematics textbooks published in the U.S. and found that only 3.7 % of the tasks in the textbook were R&P tasks and those mostly involved making and justifying claims empirically. These studies and many others sought a way of investigating the mode of reasoning in mathematics textbooks because textbooks are the main resource of teachers in planning their mathematics lessons [5].

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Following the steps of those researchers in this area, we aimed to analyze the ways of reasoning in mathematics textbooks that are currently used in five countries: Slovakia, Czech Republic, Italy, Norway, and Turkiye, as a part of a Horizon 2020 Project MaTeK[1]. We initially started with a framework that aimed to examine the effect of teachers' participation in the lesson study on the improvement of students' mathematical reasoning (for details, see Project LESSAM[2]). However, as the textbook analysis of different countries proceeded, we realized that this framework would not solely be sufficient to address the types of reasoning with alternative tasks presented in those textbooks. This directed our attention to developing an integrated framework that would encompass all types of reasoning presented in all five countries' textbooks.

Hence, in this study, we aim to present the integrated framework that we worked on to analyze the ways of reasoning in worked examples (i.e., the problems with explained solutions) in 8th-grade textbooks that are currently in use in Slovakia, Turkiye, Italy, Norway, and the Czech Republic.

2 METHODOLOGY

In this part, we will explain the framework integrating process, which follows four main steps: (1) starting with a proposed framework that focused on different types of reasoning (2) identification of R&P tasks among worked examples, (3) categorization of ways of reasoning through the proposed framework, and (4) developing new categories based on a review of existing frameworks, to cover and differentiate all types of reasoning in the worked examples.

National teams, consisting of two researchers, were established in each country, and they worked locally and internationally during the whole process of framework development. In each country, one mathematics textbook for 8th grade, which is currently in use, was selected for framework development because we believe that 8th-grade mathematics includes different types of R&P tasks from various areas of school mathematics.

2.1 Step 1: Starting with a proposed framework that focused on different types of reasoning

We initially started with a framework consisting of eight ways of reasoning used in Project LESSAM: Generalizing from specific cases (inductive reasoning); Evaluating mathematical claims (e.g., refuting through counterexamples); Developing conclusions through deductive reasoning; Reasoning by analogy (e.g., transferring the structure of manipulatives to the abstract context); Reasoning with images (e.g., decomposition of geometrical shapes in the process of justifying/proving); Evaluating the relevance of a mathematical model in a realistic situation; Making links among different representations (visual, symbolic, verbal, contextual, physical); Making predictions in stochastic situations (e.g., evaluating claims/information provided by media). This framework can be interpreted as a list of task types, activities/procedures, or areas of mathematics that mathematics teachers associate with reasoning and proof.

2.2 Step 2: Identification of R&P tasks among worked examples

Our aim was to analyze worked examples in mathematics textbooks in terms of R&P. However, not all worked examples are R&P tasks; therefore, we define R&P tasks by following two conditions:

1. It involves *a mathematical claim* that can be in the form of an answer to a (real life) word mathematics problem, a result of a "mathematization" such as an equation or graph, or a (general) mathematical statement.
2. It involves *argumentation* that supports the claim, not just a step-by-step solution to the problem using a standard/given algorithm.

[1] Enhancement of research excellence in Mathematics Teacher Knowledge Project, webpage: https://www.projectmatek.eu/

[2] Project LESSAM webpage: https://www.ucy.ac.cy/lessam/en/

2.3 Step 3: Categorization of ways of reasoning through the proposed framework

As the analysis of different countries' textbooks proceeded, we realized that the framework proposed in Step 1, although it contains very important information, would not be sufficient and efficient for our purposes: categories are not disjoint, and/or they correspond to different criteria.

2.4 Step 4: Developing new categories based on a review of existing frameworks

Based on Step 3, we sought to find (while keeping as many useful aspects as possible from the framework proposed in Step 1) disjoint categories according to one (main) criterion (core types of reasoning), while other criteria would be only supplementary. This led to a form of a two-dimensional matrix; a coding table that has ways of reasoning in rows and information regarding the use of representations (graphical, symbolic, verbal, real-world situations, manipulatives) and technology in columns (see Table 1).

In order to find appropriate categories of ways of reasoning, we have consulted several frameworks (cf. [1], [2], [4]). Based on these, we have proposed the first draft of the revised framework to encompass and differentiate all types of R&P tasks presented in all five countries' textbooks. Then, we coded the identified R&P tasks again with the revised framework. After the second coding, the framework was adjusted again, and two new categories were introduced: *simple 1-step deduction* and *mathematising*. The reason was to highlight the importance of simple deductions as the building blocks of proving [6, p. 235] and mathematising as one of the fundamental mathematical activities [7, p. 81]. At the same time, with the aim of covering all possible ways of reasoning, we kept categories *Appeal to authority* and *Other*, although they did not appear in the textbooks selected for this study.

We took several steps through the development process to increase the validity and reliability of the study. Several meetings of national teams were held to have a shared understanding of the categories in the framework and to check the coding process with task examples. In addition, national teams matched in pairs to compare and contrast the coded tasks, and they discussed until they agreed on the coding and the categories.

3 RESULTS

In this section, we present our framework and provide an example coding from analyzed textbooks.

As seen in Table 1, in the framework we aimed not only to identify the core types of reasoning but also to explore components of some types such as whether the reasoning with empirical arguments/specific cases aims to make a claim or justify a claim, or whether deductive reasoning utilized a generic example, counterexample or systematic enumeration.

Furthermore, in our framework, we aimed to understand whether different representations accompany different ways of reasoning. In line with Duval [8], we are convinced that comprehension in mathematics assumes the coordination of at least two different representations, and moreover, changing representations is the threshold of mathematical comprehension for learners at each stage of the curriculum. Therefore, as we code the ways of reasoning, we also intended to classify different representations (e.g., graphical (G), symbolic (S), verbal (V), real-world situations (R), manipulatives (M)) when applicable.

We have included column T regarding the use of (digital) technologies due to the growing importance of their use in teaching mathematics.

Table 1. An integrated framework for analysing R&P worked examples in textbooks

Way of reasoning/ specification	**R***	**T***	**RT****	**None**	**Description[3]**
1. Appeal to authority					"In *appeal to authority*, the warrant (in Toulmin's sense) given to justify an assertion is that a figure of authority (e.g., Euclid, a textbook) says it is so. From a mathematical point of view, this is no explanation or reasoning at all: perhaps it might be called a 'null-explanation.'" [1, p. 278]
2. Simple (1-step) deductive reasoning					Simple (1-step) deductive reasoning is a single deduction from one or more premises. [cf. 6, p. 235]
3. Mathematising					In our context, under mathematising we understand the explanation/ justification of transformation/ decontextualization of a word problem/ a problem defined in the real world, to a strictly mathematical form. [cf. 7, p. 81]
4. Reasoning by analogy					"Reasoning by analogy involves making a conjecture based on similarities between two cases, one well known (the source) and another, usually less well understood (the target)." [9, p. 110]
5. Reasoning with empirical arguments/specific cases: a) making claims and generalizing b) justification of a claim (extra note if experimental demonstration is used)					Reasoning begins with specific cases and produces a generalization from these cases [cf. 9, p. 88]; and testing claims using "evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, and so forth" [10, p. 809]
6. Developing conclusions/ justifying/refuting through deductive reasoning a generic example, a counterexample, a systematic enumeration[4] other					"Deductive reasoning … is the process of inferring conclusions from known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments." [11, pp. 235-236]
7. Other (abductive reasoning, etc.)					"Abductive reasoning … [is] the search for a general rule from which a specific case would follow." (Eco's description in [9, p. 101])

* ***R*** *Using at least 2 different* ***r****epresentations: Graphical (G), Symbolic (S), Verbal (V), Real-world situations (R), Manipulatives (M)*

* ***T*** *Using (digital)* ***t****echnology (e.g., calculator, GeoGebra, math apps, ...)*

** ***RT*** *Using both technology and at least two different representations together*

In the section below, we illustrate how this framework worked in our textbook analysis. Due to the space limitation, we provide only two sample coding from two textbooks, however our future work in this area will address how we used this framework to analyze 8th-grade mathematics textbooks that were structurally different in each of the five countries and what conclusions this framework allowed us to make in our comparative textbook analysis (see future publications of Project MaTeK).

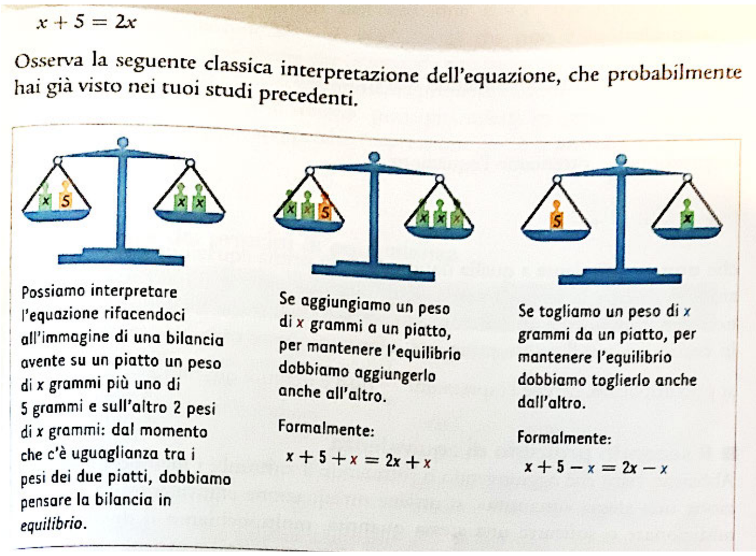
3.1 Sample Coding of Worked Examples

Our analysis of 8th-grade mathematics textbooks in five different countries showed that worked examples can be found in different forms in different sections of a textbook. In the textbooks we analyzed, worked examples were either presented as traditional exercises where a command/question and a solution/answer are provided, or blended in the content of a narrative.

[3] Our categories are identical or very similar to categories known from the literature. Therefore, when describing them, we use either a direct quote or a minor modification of the text from established sources.

[4] The solution includes justification (implicit or explicit) that all possible cases were considered.

Since the second form is rarer, both the following examples are of this type. Our first example is from an Italian 8th-grade textbook. The task in Fig. 1 discusses the equivalence principles in linear equations:


$x + 5 = 2x$

Osserva la seguente classica interpretazione dell'equazione, che probabilmente hai già visto nei tuoi studi precedenti.

Possiamo interpretare l'equazione rifacendoci all'immagine di una bilancia avente su un piatto un peso di x grammi più uno di 5 grammi e sull'altro 2 pesi di x grammi: dal momento che c'è uguaglianza tra i pesi dei due piatti, dobbiamo pensare la bilancia in equilibrio.

Se aggiungiamo un peso di x grammi a un piatto, per mantenere l'equilibrio dobbiamo aggiungerlo anche all'altro.

Formalmente:

$x + 5 + x = 2x + x$

Se togliamo un peso di x grammi da un piatto, per mantenere l'equilibrio dobbiamo toglierlo anche dall'altro.

Formalmente:

$x + 5 - x = 2x - x$

x + 5 = 2x

Observe the following classic interpretation of the equation, which you probably already saw in your previous studies.

We can interpret the equation by referring to the image of a scale having on one plate a weight of x grams plus one of 5 grams and 2 weights of x grams on the other one. There is equivalence between weights of the two plates and so the scale is in equilibrium.

If we add a weight of x grams on one scale's plate, with the aim of maintaining the equilibrium, we have to do the same on the other plate. Formally x + 5 + x = 2x + x.

If we remove a weight of x grams on one scale's plate, with the aim of maintaining the equilibrium, we have to remove the same on the other plate. Formally x + 5 - x = 2x - x.

Figure 1. Worked example from Italian textbook [12, p. 457]

The mathematical claim and argumentation that are used to identify the task in Fig. 1 as an R&P task are as follows:

1 Mathematical claim:

"5 is the solution to the equation x + 5 = 2x".

(Note that the mathematical claim is only visualized (in the last image of the scales.))

2 Argumentation:

"We can interpret the equation by referring to the image of a scale .... There is equivalence between weights of the two plates and so the scale is in equilibrium."

"If we remove a weight of x grams on one scale's plate, with the aim of maintaining the equilibrium, we have to remove the same on the other plate."

(Note that the solution, as well as the reasoning are visualized in the accompanying image of the scales.)

The mathematical claim and argumentation identified in the R&P task shown in Fig. 1 indicated *Reasoning by analogy* because the task aimed to establish a relationship between a scale model and equations, "based on similarities between two cases" [9, p. 110]. This task was also coded for involving at least two representations in the used reasoning. Since the explained solution includes a visual image of the scale model, no matter whether we classify it as a (purely) graphical representation or (mental) manipulative, together with the verbal and symbolic representations, three different representations are used. At the same time, no (digital) technology is used in this R&P task, thus the column R was used in the coding.

Our second example is also of the latter form (a blended mathematical argumentation in a narrative) and comes from a Norwegian 8th-grade textbook (see Fig. 2). This narrative serves as an introduction to how linear functions were related to practical situations.

> We will look closer to how linear functions are used to understand, describe and analyse practical situations.
>
> If you buy buns that cost 8 kr per piece, the price you will pay will be a function of how many buns you buy. The price is eight times the total number of buns. For every bun, you buy extra the price increases by 8 kr. The linear function has *slope* 8.
>
> If you do not buy any buns, you do not pay anything. Therefore, the *y-intercept* is 0. The price you pay is a function of the total number of buns you buy. We can call this price P and get the following expression:
>
> P(x) = 8x

Figure 2. Worked example from Norwegian textbook [13, p. 176]

The mathematical claim and argumentation that are used to identify the narrative in Fig. 2 as an R&P task are as follows:

1. Mathematical claim:

   "If you buy 8 buns that cost 8 kr per piece, the price as a function of the number of the buns is given by P(x) = 8x".

2. Argumentation:

   "For every bun you buy extra the price increases by 8 kr. The linear function has slope 8".

   "If you do not buy any buns, you do not pay anything. The y-intercept is 0".

   (Note that it is assumed that the students know the general form of a linear function from the previous section.)

The task is about justifying the transformation of a real-life situation to a strictly mathematical form and is therefore coded as *mathematising.* One of the main characteristics of the mathematising code is decontextualizing the context given in the problem, which requires a particular way of reasoning. For used representations, this R&P task involves mostly verbal (V) but also some symbolic (S) representation, and therefore it was coded for at least two representations. At the same time, no (digital) technology is used in this task, thus, the column R was used in the coding.

As mentioned earlier, those two examples of task coding served to illustrate how we utilized our framework for identifying ways of reasoning in mathematics textbooks. There is no doubt that each of the textbooks from five countries has similar and different tasks, and we continue our deep constant-comparative analysis. In the next section, we briefly discuss our current conclusions and future research plans regarding the use of this framework in mathematics textbook analysis.

4 CONCLUSIONS

Textbooks are considered as the important element in educational system [14] and the content of the textbooks is important in student learning [15]. The aim of this research was to develop an integrated framework that can be used for analyzing ways of reasoning in mathematics textbooks. During the process, initially frameworks that were used in the textbook analysis were searched, reviewed, and analyzed [1, 2, 4]. Then, to address all possible ways of reasoning in five different countries' textbooks, an integrated framework was created. This integrated framework was modified several times through analyzing R&P tasks given in 8th-grade textbooks that are currently used in five countries: Slovakia, Czech Republic, Italy, Norway, and Turkiye.

Analysis revealed that integrated framework consisted of 7 ways of reasoning specifications and items under those specifications further categorized whether they consisted of different modes of representations and usage of (digital) technology. Although the categories in the integrated framework already existed in the literature, our study confirmed particularly Reid and Knipping's categorization [9] that is "relevant to teaching and learning proof." Furthermore, our content analysis showed that the framework is valid and comprehensive enough to cover and categorize R&P tasks given in mathematics textbooks.

Due to the space limitation, we provided only two worked examples here, but each textbook includes different forms of tasks involving reasoning (e.g., command/question-solution, a blended narrative of the content involving reasoning). The presented integrated framework was validated by using 8th-grade mathematics textbooks used in five different countries. Although our analysis was limited to only one textbook from each country, we believe for further research that the integrated framework can be used to make comparisons among the R&P tasks located in different countries' textbooks and also at different grade levels, or for comparisons of different textbooks of the same grade within a specific country.

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REFERENCES

[1] K. Stacey and J. Vincent, "Modes of Reasoning in Explanations in Australian Eighth-Grade Mathematics Textbooks," *Educational Studies in Mathematics*, vol. *72,* no. 3, pp.271–288, 2009. http://www.jstor.org/stable/25619864

[2] N. K. Bieda, X. Ji, J. Drwencke, and A. Picard, "Reasoning-and-proving opportunities in elementary mathematics textbooks," *International Journal of Educational Research,* vol. 64, pp. 71–80, 2014. doi.org/10.1016/j.ijer.2013.06.005

[3] B. Silverman and R. Even, "Textbook explanations: Modes of reasoning in 7th grade Israeli mathematics textbooks," In *Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education (CERME9)* (K. Krainer and N. Vondrová, eds.), pp.205–212. Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME, 2015.

[4] G. J. Stylianides, "An Analytic Framework of Reasoning-and-Proving," *For the Learning of Mathematics,* vol. 28, no,1, pp. 9–16, 2008. http://www.jstor.org/stable/40248592

[5] J.T. Remillard and D.J. Heck, "Conceptualizing the curriculum enactment process in mathematics education," *ZDM Mathematics Education,* vol. 46, pp. 705–718, 2014. https://doi.org/10.1007/s11858-014-0600-4

[6] D. A. Reid, "Describing Reasoning in Early Elementary School Mathematics," *Teaching Children Mathematics*, vol. 9, no.4, pp. 234–237, 2002. http://dx.doi.org/10.5951/TCM.9.4.0234

[7] OECD, "PISA 2018 Mathematics Framework", in *PISA 2018 Assessment and Analytical Framework*, 73–95, Paris: OECD Publishing, 2019. https://doi.org/10.1787/13c8a22c-en.

[8] R. Duval, "A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics," *Educational Studies in Mathematics*, vol. 61, pp. 103–131, 2006. https://doi.org/10.1007/s10649-006-0400-z

[9] D. A. Reid and C. Knipping, Proof in Mathematics Education: Research, Learning and Teaching. City/State: Sense Publishers, 2010.

[10] G. Harel and L. Sowder, "Towards comprehensive perspectives on the learning and teaching of proof" in *Second handbook of research on mathematics teaching and learning* (F. Lester, eds.), 805–842, Reston, VA: National Council of Teachers of Mathematics, 2007.

[11] M. Ayalon and R. Even, "Deductive reasoning: in the eye of the beholder," *Educational Studies in Mathematics*, vol. 69, pp. 235–247, 2008. https://doi.org/10.1007/s10649-008-9136-2

[12] L. Sasso, Matematica a colori. Grafici, problemi e modelli. Corso di Matematica per il biennio. Algebra 1. Novara: Petrini, 2011.

[13] G. N. Tofteberg, J. Tangen, L. T. Bråthe, I. Stedøy and B. Alseth, *Maximum 8 Matematikk.* Oslo: Gyldendal Norsk Forslag, 2020.

[14] L. Haggarty and B. Pepin, "An investigation of mathematics textbooks and their use in English, French and German classrooms: Who gets an opportunity to learn what?" *British Educational Research Journal, 28*, 567–590. 2002.

[15] S. Vicente, L. Verschaffel, L., R. Sánchez *et al.,* “Arithmetic word problem solving. Analysis of Singaporean and Spanish textbooks”. *Educational Studies in Mathematics.* 2022. https://doi.org/10.1007/s10649-022-10169-x