Differences in the Comprehension of the Limit Concept Between Prospective Mathematics Teachers and Managerial Mathematicians During Online Teaching

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Abstract. The paper deals with university student's understanding of a limit process. We involved two groups of students with different specializations in the research: a group of pre-service mathematics teachers (PMTs) and a group of students of managerial mathematics (MNGs). Since the objectives for learning higher mathematics, particularly mathematical analysis, differ significantly, we expected a significant difference in reasoning between these groups. Therefore, we identified (literature- and empirical-based) the most common obstacles and misconceptions when learning a concept of limit. A teaching series was prepared, enacted, and analyzed. When analyzing students' solutions, we applied codes from the literature with minor changes. We used two-dimensional contextual analysis to work with students' answers and provided explanations: type of argument (based on Stylianides's work) and representation. Our findings confirm the problems identified in the literature when we looked at the groups without distinguishing them. Moreover, we identified and discussed some specific outcomes in the group of PMTs and MNGs separately.

Keywords: teaching and learning calculus · online learning environment · understanding limit process · COVID-19 effect on teaching calculus · interactivity during online teaching

1 Introduction

Mathematical analysis (at least on the basic level) is intertwined through all STEM-oriented university studies. It is a critical milestone in various transition processes from secondary school to university mathematics. For many students it is "the necessary evil" to pass through, and many of them struggle in their courses. One reason of this struggling could be the huge difficulty of linking between the knowledge of mathematics learned at university and the knowledge acquired in secondary school. This gap between mathematical levels and institutional cultures can lead to several study problems of freshmen.

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As Pinkernell [1] summarizes, students meet different level of rigor in communication or reasoning, and institutional differences, e.g., concerning the didactic of teaching and learning mathematics. The other problem could be that situation at some study programs is almost the same as one of the interviewees said in research made by Bosch et al. [2], "…the exercises is a list that comes from father to son. It's the same list that has been there for the past 10 years. […] the key for 60% or 70% of the students to pass is to do an exam that is not essentially different from previous one". Critical reason could be also that many students are only passive listener and users of calculus. The algorithmic characteristic of the tasks solved in the lessons and tests can lead to passing through exams without deeper understanding on subject matter. In terminology of Boaler and Andrew-Larson [3], most of our students have "received knowing", which means, they believe that doing mathematics means to memorize and quickly recall information needed.

2 Objectives and Literature Review

Limit concept appeared in early times especially in connection with determining the measure of shapes with curvilinear sides (for example, Archimedes used this concept to calculate the area of the circle and measures of another shapes and solids). A similar use of the limit process also appears in the works of mathematicians of the 17th century (e.g., Kepler, Cavalieri), the use of limit concept with the current meaning of derivative we can find in works of Descartes and Fermat. Dutch engineer Simon Stevin and Italian mathematician Luca Valerio use the concept of limits to replace the need for a double reductio ab absurdum in the ancient Greek method of exhaustion. It was only about 150 years later that the rigorous definition of the limit was constructed through the works of Cauchy and Weierstrass [4]. Cauchy build on d'Alembert understanding of the limit concept and used this notation in defining of concepts as derivative and continuity. He used infinitesimally small quantity with sufficient refine as a variable with zero limit [5, 6]. Weierstrass substituted dynamical concept of the limit based on the phrases like "approaches" or "arbitrarily close" with static one, which is known as "epsilon-delta" definition of limit. The definition (1) is one of the most common definitions of the limit which students at the STEM-oriented universities are comping with (also called "epsilon-delta" definition)

$$\lim_{x \to a} a_n = L \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; |x - a| < \delta \Rightarrow \; |f(x) - L| < \varepsilon \tag{1}$$

Students' understanding to the formal definition (1) requires students to decode its meaning from a relatively complex symbolic statement. A slight change in the wording of the definition brings a great change in its mathematical content. Many students do not notice it.

Using the limit concept other fundamental concepts like continuity, differentiability and integrability are all established. The misinterpretation or incorrect using causes later problems, as shown on Fig. 1.

There are several closely related and derived concepts in differential and integral calculus, series, etc. As studies (e.g., [8–13]) or the practical experiences of educators confirm, students can master many formalized rules and computational algorithms of

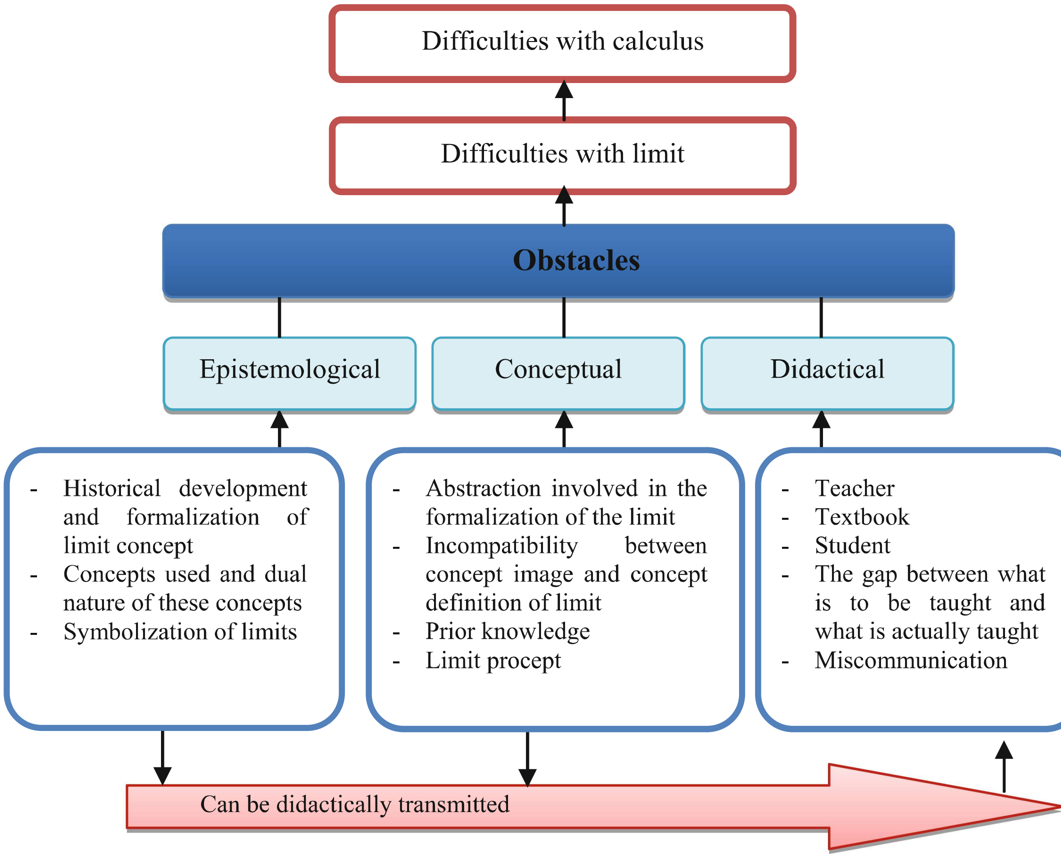

Fig. 1. Representation of limit-related obstacles (source: [7])

mathematical analysis, but at the same time a deeper understanding of the problem (conceptual understanding) may not occur.

Most studies dealing with the mentioned problem states the well-known fact that understanding the concept of limit causes students' considerable problems (e.g., [9, 14–18]). Williams [15] even claims that a complete understanding of the concept of limit is very rare even in the case of students in the first two years of university studies. Indeed, Williams [15] agrees with the opinion of Ervynck [14] that most students will not fully understand the concept of limit even after completing a calculus course (either in the case of limits of a sequence or limits of a function) and their idea of this concept does not correspond to the formal definition of limits. A similar conclusion was reached by Bastürk and Dönmez [19], who investigated the understanding of the concept of limits among pre-service teachers.

Cornu [16] pointing on that a limit concept is the first topic in which students meet infinity, or infinitesimal process in explicit form. Orton [20] declare that students understand infinity as extremely big number. Consequence of this understanding is using symbol ∞ as representant of variable or notation of number (as in algebra). Sierpinska [11] observed the same phenomenon, when students use the analogy: "since $\frac{k}{k} = 1, k \in N$ then $\frac{\infty}{\infty} = 1$."

Based on our previous research, [21], freshmen are struggling with basic properties of functions (mostly goniometric and logarithmic), have no experiences with formal mathematical notation and therefore are not able to follow the lectures. Moreover, they have problems with logical structure of statements containing several quantifiers and lack experience with rigor in reasoning. Therefore, in our research we focused on connected knowledge in terms of reasoning and proof in the two groups of students with different study programs at Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Slovakia.

3 Methodology

3.1 The Sample and Stating of the Research Questions

We have chosen two groups of students: pre-service mathematics teachers (PMTs) and group of managerial mathematics students (MNGs). In the Table 1 are the characteristics of the groups.

Table 1. Group characteristics.

	PMTs group	MNGs group
Number of students	24	28
Expected knowledge	Basic knowledge of mathematical disciplines with the aim on overview of mathematics from higher level	Extensive knowledge of mathematical disciplines focusses on applicability in the field of management, economy, and finance
Expected outcome	Conceptual understanding	Procedural fluency (with conceptual understanding)
Goal of the study program	Knowledge about how to express math ideas using mathematical language and symbolism	Mathematical modelling (in context of management, economy, and finance)
Lessons per week	2 (lectures)/2 (practical)	4 (lectures)/4 (practical)

As can be observed in Table 1, there are different expected knowledge, outcomes (based on [22]) and goals of the study program. While for PMTs the course of higher mathematics is kind of superstructure over the high-school mathematics, for MNGs it is a tool for modelling. And this is what influenced all the activities on the lessons when working with these groups.

We observed the lessons with changed approach to teaching a limit process and tested two groups of students at our faculty on comprehension of the definition of a limit of a sequence (how can small changes, e.g., order in quantifiers, in definition of specific concept influence the meaning of the definition) through ability to give a valid argument, justification.

When analyzing the students' solutions, we kept in mind our research question: *How the ways of reasoning in the groups of PMTs and MNGs differ when teaching/learning in online environment?*

3.2 Preparation and Learning Phase: Lessons Descriptions

Due to COVID-19 lessons had to be shift into online environment. We were aware of the fact, that for first-semester mathematical analysis students the notion of limit is key and, at the same time, the most difficult concept to grasp. This concept needs students to overcome significant cognitive obstacles, which are necessary to understand fully the classic "$\varepsilon - n$" (the simpler case) or "$\varepsilon - \delta$" definition of a limit.

When designing the teaching sequence focused on sequence convergence, we understood the students' need for numerous experiences with infinite sequences to be able systematically to mathematize their experiences and developed mental schemas for classifying given sequences as eighter convergent or divergent. With the students, we discussed various real situations in which the limit process can be used (e.g., the level of the drug in the patient's blood for a long time, finding the circumference of a circle, etc.).

We provided and encouraged the students to work with geometrical representations of the limit process [23] or a graphic representation of the definition of limits of sequence (ε -stripes) by using digital technology (Fig. 2) and observe how and why the formal definition works. To eliminate the most common misconceptions concerning a limit of a sequence, we provided a set of examples that were (in most cases) in contradiction with students' prior incorrect understanding of this notion.

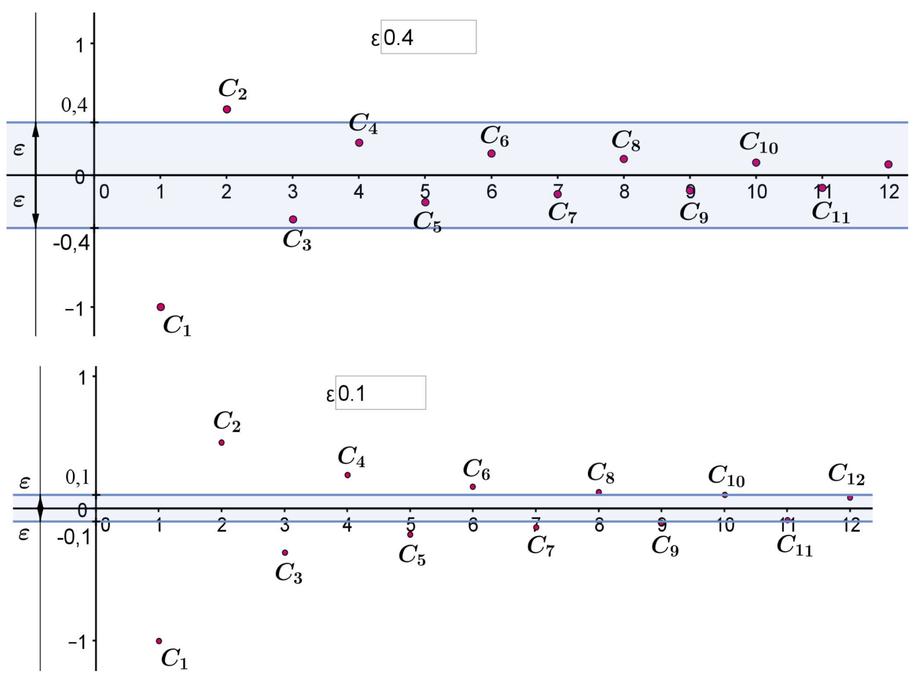

Fig. 2. Epsilon stripes for specific values of ε

For example, by utilization of divergent sequence $\{a_n\}_{n=1}^{\infty} \equiv \left\{-\frac{1}{2}, \frac{1}{n}\right\}_{n=1}^{\infty}$ students could realize that to describe the behavior of the convergent sequence is necessary universal quantification of ε, upon which n_0 is dependent and not vice- versa (Fig. 3).

In the next two examples we are providing some epistemological obstacles we tried to avoid when working with limits on the lessons.

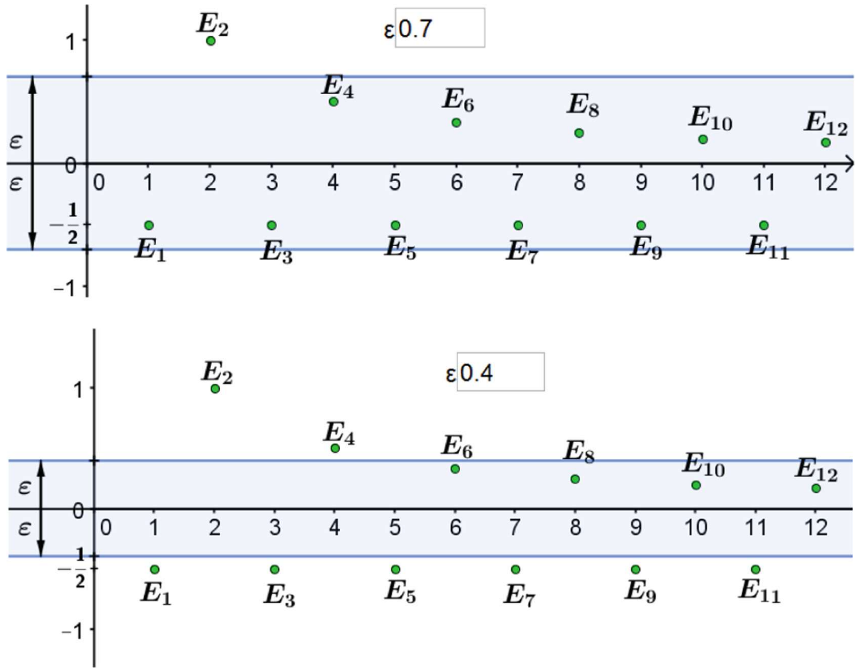

Fig. 3. Epsilon stripes used on divergent sequence $\{a_n\}_{n=1}^{\infty} \equiv \left\{\frac{1}{2}, \frac{1}{n}\right\}_{n=1}^{\infty}$

Limit as an Asymptote

The student's interpretation is based on idea that terms are closer to specific value from one direction, but never reach or overleap it. To contradict students' misconception, or to avoid it, we can use, for example, the sequence $\left\{\frac{1}{n} + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ (Fig. 4). This sequence can be in contradiction with students' incorrect prior concept and rephrasing the definition in a sense like "the terms of sequence $\{a_n\}_{n=1}^{\infty}$ are closer and closer to its limit L when n is bigger, but they never reach L".

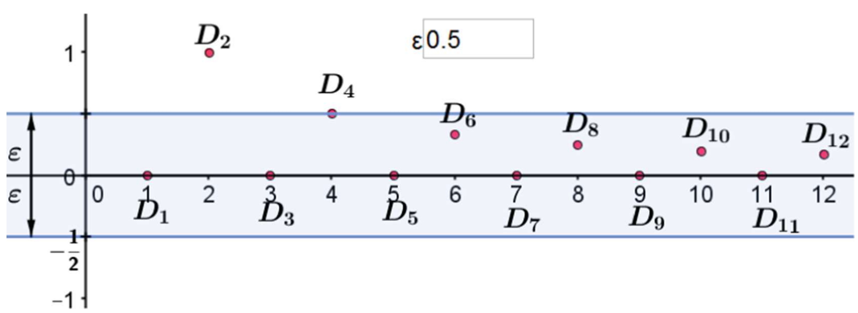

Fig. 4. Graphical representation of sequence $\left\{\frac{1}{n} + \frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

In the context of monotone bounded sequence, we discussed with the students whether the following statement is correct: "Number L is called the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ if its terms are closer and closer to the number L when n is bigger."

Consequently, we discussed with students whether condition $L - a_n > L - a_{n+1}$ (in the case of sequence increasing and bounded above) for every $n \in N$; $n > n_0$, uniquely determines the only number L, that is limit of that sequence.

Limit as Accumulation Point of Sequence

In the meaning when students change the correct definition of limit of sequence with following statement: Real number L is called the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ if for every $\varepsilon > 0$ there exist an $n_0 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for infinitely many $n > n_0$.

Even in this case can be the previously mentioned sequence $\{\boldsymbol{a_n}\}_{\boldsymbol{n}=1}^{\infty} \equiv \left\{\frac{1}{2}, \frac{1}{\boldsymbol{n}}\right\}_{\boldsymbol{n}=1}^{\infty}$ (Fig. 3) create cognitive dissonance in the student's mind as it could lead to understanding that his prior concept is incorrect.

3.3 Testing Phase

In the test we focused on the type of argument elicited (in terms of Bieda et al. [24]), different types of arguments (empirical argument, counterexample, formal proof, etc.) and using different representations (e.g., graphical, symbolical, verbal, etc.).

We gave students in both groups the test a semester later from two reasons:

1. To find out whether their knowledge is connected in a sense of [3] (in other words, we wanted to prevent using recitation of the formulas without understanding)
2. To see the effect of online teaching during the pandemic COVID-19

Both groups of students were given the same test with tasks like the ones we worked with and discussed. The test comprises one task focus on justification claims, more specific on decision which claim is equivalent to the (well know) definition of a limit of sequence. The chosen task for the students was as follow:

Which of the following formulations is equivalent to the correct definition of the sequence limit? Justify your statement in detail (indicate which part of the statement contradicts the formal definition or give an example that points to this contradiction).

a) A real number $a \in \mathbb{R}$ is a limit of a sequence $\{a_n\}_{n=1}^{\infty}$ if and only if for every real number $\varepsilon > 0$ exists finite subset $\mathcal{M} \subset \mathbb{N}$ so for every $n \in \mathbb{N} \backslash \mathcal{M}$ stands $|a_n - a| < \varepsilon$.
b) A real number $b \in \mathbb{R}$ is a limit of sequence $\{b_n\}_{n=1}^{\infty}$ if and only if for every real number $\varepsilon > 0$ exists natural number $n_0 \in \mathbb{N}$, so for infinitely many $n > n_0$, $n \in \mathbb{N}$, $n \in \mathbb{N}$ stands $|b_n - b| < \varepsilon$.
c) A real number $c \in \mathbb{R}$ is a limit of sequence $\{c_n\}_{n=1}^{\infty}$ if and only if there exists such real number $\varepsilon > 0$, so for every natural $n > n_0$, $n \in \mathbb{N}$ stands $|c_n - c| < \varepsilon$.

Students had 20 min to decide these three "definitions" and support their decision with arguments, justification. Since we postpone the testing phase in one semester, the test was written in the classroom (face-to-face lesson).

3.4 Analysis of the Obtained Data

When analyzing the students' solutions, we kept in mind our research question: *How the ways of reasoning in the groups of PMTs and MNGs differ when teaching/learning in online environment?*

We used adjusted framework based on Bieda et al. [24] and Stylianides [25] to create categories for different solving strategies. Since we tested only "justifying claims" (in terms of Bieda et al. [24]), the adjusted framework had the structure presented in Fig. 5.

Empirical argument is an argument that provides inconclusive evidence for the truth of mathematical claim (similar to [26] "empirical justification" and [27] "naive empiricism"). *Rationale* - capture arguments for or against mathematical claim that are neither proofs nor empirical; "transition between empirical reasoning and proof-type reasoning." This type of argumentation is not sufficient in mathematics.

Generic example is an argument that uses a particular case seen as representative of general case (similar to [26] "transformational proof"). *Demonstration*, according to [25], this term represents argument does not rely on the representatives of particular case (similar to [26], "axiomatic proof" and [27], "thought experiment"). This includes valid deductive arguments by counterexample, contradiction, mathematical induction, contraposition, exhaustion, etc.

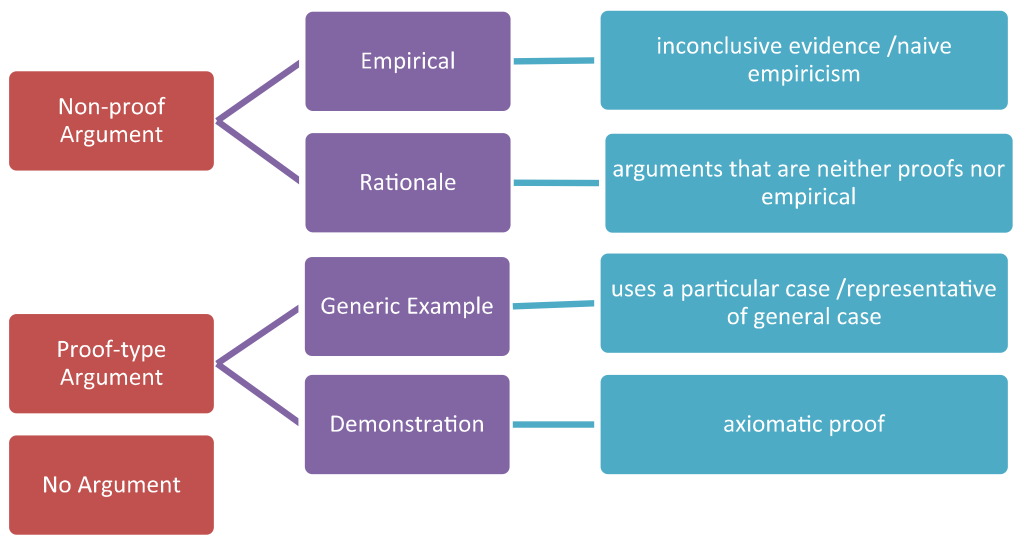

Fig. 5. Adjusted frameworks used for the analysis of students' solutions in justifying claims

We divided solutions into the groups with the same characteristic according to the used argument, in the following phase we made subgroups based on the representation used, like presented in Table 2.

For each category in the first 2 columns there are 5 different representations to be used by students when solving the tasks. Therefore, we expected 5 groups and 5 subgroups for each group.

Table 2. Expected students' solution strategies.

Argument type	Additional type of given argument	Used representation
1. Non-proof	empirical	a. algebraic b. figural c. verbal d. switch between representations e. other
	rationale	
2. Proof	counterexample	
	demonstration	
3. no argument		

4 Results and Discussion

4.1 Pre-service Mathematics Teachers Group Results

When analyzing the solutions of PMTs, we identified several types of justifications, or explanations of their thinking (see Appendix 1). In that summary, we did not distinguish correctness or incorrectness of the provided solution by PMTs. Since we looked for every PMTs and each part (a-c) on the type of argument and representation, the sum of the numbers in the Appendix 1 is tripled.

Looking on the correctness of the solution and provided arguments, there were only 3 students who answered correctly (that option "a" is an equivalent definition to the original one and provided counterexamples to the options b and c. In the rest of the group, we identified epistemological obstacle "limit as an accumulation point", two conceptual obstacles: (i) overgeneralization, when students manipulated with infinite sets like with finite sets, and (ii) use/understood "for infinitely many" as equivalent of "for all" (see Fig. 6, answer b).

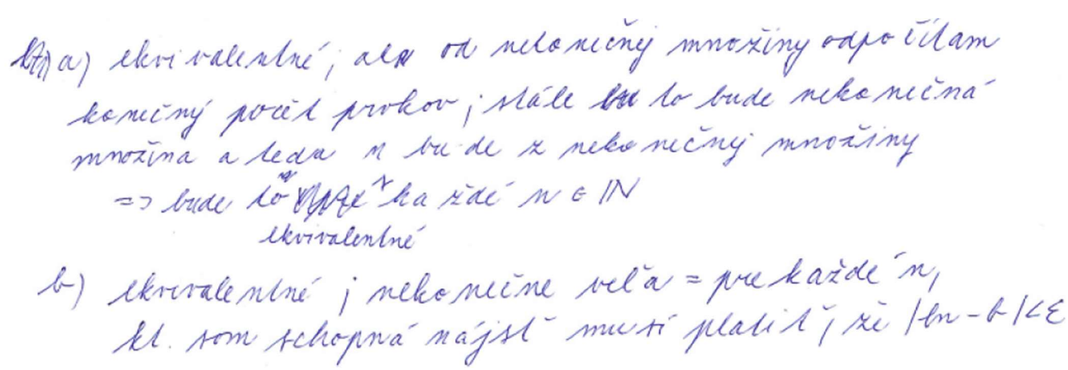

Fig. 6. Example of PMT solution [translation: a) equivalent, if we subtract finite number of members from infinite set, this set remains infinite, therefore n belongs to the infinite set => it will be equivalent for any natural n] [translation: b) equivalent; infinitely many = for any n, which I can find must stands $|b_n - b| < \varepsilon$]

Since we try to prevent the last-mentioned misconception (ii) during classes, by providing a set of examples and tasks that could be in contradiction with PMTs prior concept, our findings showed how durable and resistant this misconception is. PMTs used mostly proof-type arguments with algebraic and verbal representation for their arguments, or switch between algebraic, verbal and symbolic.

4.2 Managerial Mathematics Students Group Results

In the case of MNGs students, the results were similar, deductive arguments by counterexample or attempt to produce generic example prevailed among them as well. Summary of MNGs working is in Appendix 2.

Looking on a correctness of MNGs justification, only one student correctly identified formulation equivalent with proper definition of limit of sequence and correctly reasoned why the remaining two formulations are not equivalent. The remaining students marked as the correct answer formulation option b (Fig. 7) with very similar misconceptions as were identified in group of PMTs. The case in Fig. 7 points on use/understood "for infinitely many" as equivalent of "for all".

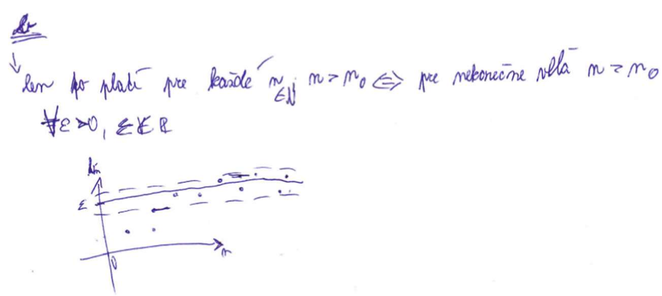

Fig. 7. Example of MNG's argument [Translation of the text: b [is correct] but it works for every natural $n \in \mathbb{N}$, $n > n_0 \Leftrightarrow$ for infinitely many $n > n_0$, for any real $\varepsilon > 0$]

In many cases was clearly identified memorized second part of the definition (Fig. 8) This phenomenon is often observe in students on applied study programs not only by us.

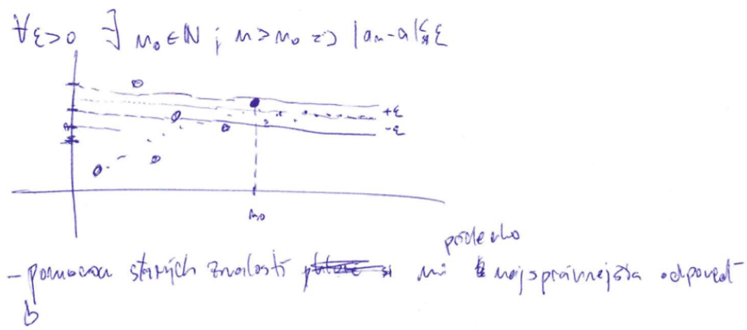

Fig. 8. Example of imitating reasoning [translation: I use the old knowledge to find out the correct solution – b]

In both examples in Fig. 7, and Fig. 8 are visible imitating what was done on the lessons (e.g., epsion-stripes, remembered part of the formula) besides the described misconceptions. MNGs used also proof-type arguments but mostly figural representations of their arguments or switch between figural and verbal.

4.3 Discussion

Several studies (e.g., [15, 28, 29]) confirm our results and experiences. The limit concept is difficult to students since statical definition of the limit (sequence or function) is in contradiction with their intuitive dynamic understanding of this process. When describing a limit process or reading symbolical representation ($\lim_{n\to\infty} a_n = L$, $\lim_{x\to a} f(x) = L$), students use expressions like "with bigger n sequence members are closer to number L", "when x approaches to a values of the function f are infinitesimally close to the value L". These expressions have a dynamical character.

As we observed, transformation of dynamic form of the limit to the statics symbolic expression using quantifiers could be challenging for students in both groups. Several misconceptions were identified. The most common were epistemological obstacles, when students applied properties of finite sets to the infinite ones (as described in [8]), problems caused by fundamental linguistic flaws in the standard presentation of limit (as identified in [4]) and misunderstanding of the quantifier logic in mathematical statements.

Students often use imitative reasoning, e.g., copy algorithms or recall facts, when solving mathematical tasks [30]. As we demonstrated in Fig. 8, students memorize the formula as a picture without deeper understanding it.

The data were collected at a time when the students were already familiar with the terms and concepts defined using the term sequence limit, or functions (derivative of a function, a definite integral, the sum of an infinite series). Our findings show that if the basic concept is not correctly understood, additional mathematical superstructure and more experience with the mentioned concepts do not guarantee that there will be a correction and a deeper understanding of the basic concepts (e.g., [31]) Also, for this reason, it would be worth considering devoting a certain amount of time so that students arrive at the formulation of the correct definition of limits on their own with the help of guided research.

Some problems could be caused by online teaching due to the COVID-19 pandemic. Missing spontaneous discussion between teacher and students when working with new concepts, instant feedback from facial students expressions, and lack of peer interaction outside the classroom were the most common problems we observed. On the one hand, students appreciated that lessons (both lectures and practical) were recorded, so they could watch them again when they missed something. On the other hand, PMTs and MNGs pointed to missing group learning in dormitories or study rooms.

5 Conclusion

Even though two studied groups of students had different backgrounds and mathematical training at the university, there are no significant qualitative differences between these groups when answering our questions. On the other hand, we observed higher effort to reason and prove the answers in the group of MNGs.

The most common argument in both groups was by providing a counterexample. Although students during the semester encountered several different representations of sequences and their limit (graphical, algebraical, numerical, topological), the most popular way of solving the tasks was figural, by using epsilon stripes.

We observed no significant differences between the groups, therefore, more targeted lessons are needed. Lessons should be more connected to the practice (e.g., economy, sciences), this implies cooperation among different departments (mathematics, sciences,…), which can help to provide a set of examples that will be in contradiction with student prior understanding of the problematic's notion.

The second outcome is that working on deeper understanding of mathematical notation and language is needed. Students on lower levels should work more with quantifiers, changes in order or type etc. Tasks (from the lowest grades) should be focus on "making and justifying claims" (not only on justifying),

Identified obstacles are mostly epistemological (accumulation point) and conceptual (overgeneralization, misunderstanding of the concept itself). Most of the students usually memorize definitions without fully understanding them. The biggest and resisted problem in mathematical analysis observed in both groups of students was incorrect equivalence: "for infinitely many" = "for all".

All of mentioned problems could lead to didactical obstacles, when (especially) PMTs may misinterpret the role of mathematics, importance of reasoning and proof tasks, and most importantly their teaching style could be negatively influenced, and the problem could be even worse in next 10 years.

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Appendix 1

(See Table 3)

Table 3. Used type of arguments and representations by PMTs

Type of used argument	Used representation	Subtask	Number of PMTs
No argument		a	1
		b	4
		c	–
Non-proof: rationale	switch between algebraic and verbal	a	2
		b	3
		c	1
	verbal	a	2
		b	1
		c	1
Non-proof: empirical	switch between algebraic and verbal	a	0
		b	0
		c	1
Proof: counterexample	algebraic	a	0
		b	1
		c	9
	figural	a	0
		b	2
		c	1
Proof: demonstration	verbal	a	6
		b	1
		c	4
	switch between algebraic, verbal, and figural	a	1
		b	4
		c	1

Appendix 2

(See Table 4)

Table 4. Used type of arguments and representations by MNGs

Type of used argument	Used representation	Subtask	Number of MNGs
No argument		a	3
		b	4
		c	1
Non-proof: rationale	switch between algebraic and verbal	a	4
		b	5
		c	1
	switch between figural and verbal	a	4
		b	6
		c	1
Non-proof: empirical	switch between algebraic and verbal	–	–
Proof: counterexample	algebraic	a	0
		b	2
		c	13
	verbal	a	4
		b	0
		c	2
Proof: demonstration	verbal	a	2
		b	5
		c	1
	switch between algebraic, verbal, and figural	a	5
		b	6
		c	2

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